Travaux Dirigés: Semestre-S3- Série I

Groupe B-Electricité 2

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1 Exercice 1

On donne le vecteur position[1]

$$\vec{r} = x\ \vec{e}_x + y\vec{e}_y + z\vec{e}_z = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{R}^3$$

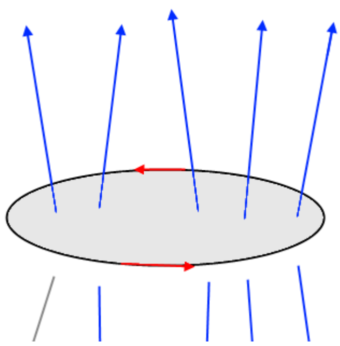

Figure 1: flux d'un vecteur à travers une surface S de bord $C = \partial S$

Question: 1) calculer

$$\text{i) } \overrightarrow{rot}\ \vec{r} = \vec{\nabla} \wedge \vec{r} \quad , \quad \text{ii) div}\, \vec{r} = \vec{\nabla}.\vec{r} \quad , \quad \text{conclure}$$

i) calcul de $\vec{\nabla} \wedge \vec{r}$

$$\vec{\nabla} \wedge \vec{r} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{\nabla} \wedge \vec{r} = \vec{0} \Rightarrow \iint_S (rot\ \vec{r})\,.\overrightarrow{dS} = \oint_{\partial S} \vec{r}.\overrightarrow{dl} = 0 \Rightarrow \vec{r} \text{ est à circulation conservative}$$

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ii) calcul de $\vec{\nabla}.\vec{r} = \text{div}\,\vec{r}$

$$\vec{\nabla}.\vec{r} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0$$

$\vec{\nabla}.\vec{r} \neq 0 \Rightarrow$ le champ $\vec{r}$ n'est pas à flux conservatif

2) Trouver la fonction $f\,(x, y, z)$ telle que $\vec{r} = \text{grad}\, f = \vec{\nabla} f$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \quad \Rightarrow \quad f = \frac{1}{2}\,(x^2 + y^2 + z^2) + \text{const}$$

3) circulation de $\vec{r}$ d'un point A à un point B de 3 facons différentes, voir fig 2

on a

$$\vec{r}.\overrightarrow{dl} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = xdx + ydy + zdz = 0 + ydy + zdz$$

voir fig 2

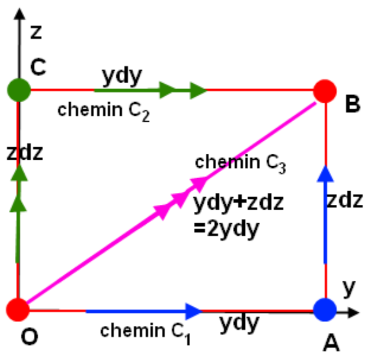

Figure 2:

$$\begin{array}{ll} \text{OAB:} & \int_{C_1} \vec{r}.\overrightarrow{dl} = \int_0^a ydy + \int_0^a zdz = \frac{a^2}{2} + \frac{a^2}{2} \\ \text{OCB:} & \int_{C_2} \vec{r}.\overrightarrow{dl} = \int_0^a zdz + \int_0^a ydy = \frac{a^2}{2} + \frac{a^2}{2} \\ \text{OB:} & \left\{ \begin{array}{l} \int_{C_3} \vec{r}.\overrightarrow{dl} \\ z = y \end{array} \right. = 2\int_0^a ydy = 2\frac{a^2}{2} \end{array}$$

aute methode

$$\begin{aligned} \int_{C_3} \vec{r}.\overrightarrow{dl} &= \int_{0 \leq z = y \leq a} (xdx + ydy + zdz) \\ &= \int_{0 \leq z = y \leq a} d\left(\frac{x^2+y^2+z^2}{2}\right) = 2\int_0^a d\left(\frac{y^2}{2}\right) \end{aligned}$$

4) flux Φ de $\vec{r}$ à travers les faces d'un *cube* de coté a et de volume a^3

$$\begin{aligned}
\Phi = \iint_{\partial V} \vec{r}.\overrightarrow{dS} \quad & \text{th de Green Ostogradsky} \qquad \Rightarrow \\
& = \iiint_V \operatorname{div} \vec{r}\, d\tau = 3 \iiint_V d\tau \\
& = 3a^3
\end{aligned}$$

1.1 Exercice 2

On donne deux points P et P':

$$\begin{array}{lll}
P\,(x, y, z)\,, & ,\quad P'\,(x', y', z') & ,\quad \vec{r} = \overrightarrow{P'P} \\
\vec{r} = \begin{pmatrix} x - x' \\ y - y' \\ z - z' \end{pmatrix} & ,\quad \vec{\nabla}_{P'} = \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} & ,\quad \vec{\nabla}_P = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}
\end{array}$$

Nous avons

$$\begin{aligned}
\vec{r}^2 &= (x - x')^2 + (y - y')^2 + (z - z')^2 \quad = r^2 \\
\tfrac{1}{r} &= \tfrac{1}{\sqrt{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]}}
\end{aligned}$$

on a aussi

$$\begin{aligned}
\tfrac{\partial}{\partial x}\left(\tfrac{1}{r}\right) &= \tfrac{\partial}{\partial x}\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-\frac{1}{2}} \\
&= -\tfrac{1}{2} \times 2\,(x - x')\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-\frac{3}{2}} \\
&= -\tfrac{1}{2} \times \tfrac{2(x-x')}{r^3}
\end{aligned}$$

soit

$$\begin{aligned}
\tfrac{\partial}{\partial x}\left(\tfrac{1}{r}\right) &= -\tfrac{1}{2}\tfrac{2(x-x')}{r^3} &= -\tfrac{(x-x')}{r^3} \\
\tfrac{\partial}{\partial x'}\left(\tfrac{1}{r}\right) &= -\tfrac{1}{2}\tfrac{-2(x-x')}{r^3} &= +\tfrac{(x-x')}{r^3}
\end{aligned}$$

Question a)

$$\begin{aligned}
\vec{\nabla}_{P'}\left(\tfrac{1}{r}\right) = \; &= \begin{pmatrix} \frac{\partial}{\partial x'}\left(\frac{1}{r}\right) \\ \frac{\partial}{\partial y'}\left(\frac{1}{r}\right) \\ \frac{\partial}{\partial z'}\left(\frac{1}{r}\right) \end{pmatrix} = \begin{pmatrix} \frac{(x-x')}{r^3} \\ \frac{(y-y')}{r^3} \\ \frac{(z-z')}{r^3} \end{pmatrix} \\
&= \tfrac{1}{r^3}\begin{pmatrix} (x - x') \\ (y - y') \\ (z - z') \end{pmatrix} = \tfrac{\vec{r}}{r^3} = \tfrac{\vec{e_r}}{r^3}
\end{aligned}$$

Question b)

$$\vec{\nabla}_P\left(\frac{1}{r}\right) = \quad = \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\ \frac{\partial}{\partial y}\left(\frac{1}{r}\right) \\ \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \end{pmatrix} = \begin{pmatrix} -\frac{(x-x')}{r^3} \\ -\frac{(y-y')}{r^3} \\ -\frac{(z-z')}{r^3} \end{pmatrix}$$
$$= -\frac{1}{r^3}\begin{pmatrix} (x-x') \\ (y-y') \\ (z-z') \end{pmatrix} = -\frac{\vec{r}}{r^3}$$

Question c): $\vec{\nabla}.\vec{r} = 3$

En coordonnées cartesien de base (e_x, e_y, e_z)

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad , \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \quad \vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

corrdonnées sphérique de base $(e_r, e_\theta, e_\varphi)$

$$\vec{A} = \begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix} \quad , \quad \vec{r} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \quad , \quad \vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{pmatrix}$$

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\vec{\nabla}.\vec{r} = \frac{1}{r^2}\frac{\partial\left(r^2 \times r\right)}{\partial r} = 3$$

Question d) flux Φ de $\vec{r}$ à traver une sphère de centre O et de rayon r

$$\begin{aligned} \Phi &= \iint_{\text{sphère: } (O,r)} \vec{r}.\overrightarrow{dS} \\ &= \iint_{\text{sphère: } (O,r)} (\vec{r}.\overrightarrow{e_r})\, dS \\ &= r\iint_{\text{sphère: } (O,r)} dS = 4\pi r^3 \end{aligned}$$

1.2 Exercice 3

Soient f est une fonction scalaire et $\vec{A}$ un champ de vecteur

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad , \quad f\vec{A} = \begin{pmatrix} fA_x \\ fA_y \\ fA_z \end{pmatrix}$$

Question a) on a:

$$\begin{aligned}\vec{\nabla}\left(f\vec{A}\right) = \operatorname{div} f\vec{A} \quad &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} . \begin{pmatrix} fA_x \\ fA_y \\ fA_z \end{pmatrix} \\ &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\ &= f\vec{\nabla}.\vec{A} + \vec{A}.\vec{\nabla} f\end{aligned}$$

Exemple: $f = x^2 + y^2 + z^2$

$$\vec{A} = \begin{pmatrix} 3x \\ y \\ 2z \end{pmatrix} \quad , \quad f\vec{A} = \begin{pmatrix} 3x\left(x^2+y^2+z^2\right) \\ y\left(x^2+y^2+z^2\right) \\ 2z\left(x^2+y^2+z^2\right) \end{pmatrix}$$

$$\vec{\nabla}\left(f\vec{A}\right) = \operatorname{div}\left(f\vec{A}\right) = 12x^2 + 8y^2 + 10z^2$$

au point $(x, y, z) = (2, 2, 2)$,

$$\operatorname{div} f\vec{A} = 120$$

b) on a $f = x^2 + y^2 + z^2$

$$\vec{\nabla} f = \operatorname{grad} f \quad = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \cdots = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

$$\operatorname{div}\vec{A} = \vec{\nabla}.\vec{A} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} . \begin{pmatrix} 3x \\ y \\ 2z \end{pmatrix} = 3 + 1 + 2 = 6$$

c) unités de mesure: $[\text{distance}] = metres \equiv m$

$$\begin{aligned} &[f] \sim [x^2] && \sim m^2 \\ &\left[\vec{A}\right] \sim [x] && \sim m && \Rightarrow \quad \left[\operatorname{div} f\vec{A}\right] \sim m^2 \\ &\left[\vec{\nabla}\right] \sim \left[\tfrac{1}{x}\right] && \sim m^{-1} \end{aligned}$$

1.3 Exercice 4

Trois relations utiles

$$\begin{aligned} \oint_{\partial S} f\ \overrightarrow{dl} \quad &= -\iint_S \overrightarrow{\operatorname{grad}} f \wedge \overrightarrow{dS} \\ \iint_{\partial V} f\ \overrightarrow{dS} \quad &= \iiint_V \overrightarrow{\operatorname{grad}} f\ d\tau \\ \iint_{\partial V} \vec{A} \wedge\ \overrightarrow{dS} \quad &= -\iiint_V \vec{\nabla} \wedge \vec{A}\ d\tau \end{aligned}$$

Exemple: Démontrons la *Formule de Kelvin*

$$\oint_{\partial S} f \ \overrightarrow{dl} = -\iint_S \overrightarrow{\mathrm{grad}} f \wedge \overrightarrow{dS}$$

Mulitiplions le terme de droite scalairement par un vecteur $\vec{e}$ consant quelconque

$$\begin{aligned}
\vec{e}.\iint \left(\vec{\nabla} f \wedge \overrightarrow{dS}\right) &= -\iint_S \overrightarrow{dS}.\left(\vec{\nabla} f \wedge \vec{e}\right) \quad , \qquad \vec{U}.\left(\vec{V}\wedge\vec{W}\right) = \vec{W}.\left(\vec{U}\wedge\vec{V}\right) \\
&= -\iint_S \overrightarrow{dS}.\left(\vec{\nabla} \wedge f\vec{e}\right) \quad \text{car} \quad \vec{\nabla}\wedge f\vec{e} = \vec{\nabla} f \wedge \vec{e} + f \underbrace{\vec{\nabla}\wedge\vec{e}}_{=0} \\
&= -\iint_S f\vec{e}.\overrightarrow{dl} \\
&= -\vec{e}.\oint_{\partial S} f\overrightarrow{dl}
\end{aligned}$$

Pour établir les 2 autres, noter que

$$\begin{aligned}
\mathrm{div}\, f\vec{A} &= f\vec{\nabla}.\vec{A} + \vec{A}.\vec{\nabla} f \\
\mathrm{div}\left(\vec{A}\wedge\vec{B}\right) &= \vec{B}.\left(\vec{\nabla}\wedge\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\wedge\vec{B}\right)
\end{aligned}$$